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A STUDY OF THE PHYSICS OF PERTURBATION OF SUBSONIC CIRCULAR JETS

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ABSTRACT

Small departures from steady flow of a subsonic circular jet in an incompressible inviscid fluid are described in terms of a series of wave mode functions. The potential of a typical component mode is:

$$V = CR_s(\ (\alpha + i\beta)r)H_s(\Theta)\exp(int + (\alpha - i\beta)z)$$

This represents the velocity potential in cylindrical coordinates with z measured along the jet axis. C is an arbitrary constant, R_s, a "modified" Bessel function representing I_s inside the jet or K_s outside. H_s is a unit amplitude periodic function of Θ. On the jet surface this represents a series of s longitudinal ridges modulated sinusoidally along it and moving downstream, or s spiral ridges or both.

The instability of columnar flow is described by the growth in amplitude along the jet. The stabilizing influence of viscous forces in real jets results in disintegration into terminal turbulence at a much slower rate than would be inferred from the idealized theory. The analysis is carried to a point where, except for certain incomplete portions pointed out in the text, it is suggested that further advance looking to a definitive rationalization of this phenomenon in real jets, is most economically made with the help of experiment. Theoretical and experimental wave transmission techniques are applicable.

In suitably controlled experiment, the structure of the terminal turbulent field is visibly dominated by the characteristics of the perturbation.

A Study of the Physics of Perturbation of Subsonic Circular Jets

1. Introduction

When an air jet is observed by means of smoke introduced into it, the pattern of flow may be found to vary considerably from one experiment to another. Under circumstances which will be defined, any departure from steady flow may be described as the sum of a series of orthogonal wave mode functions. This report contains an outline description of such functions and of their properties. On account of the difficulties otherwise introduced, the fluid will be assumed incompressible and inviscid. The limitations of this assumption will be commented on briefly.

The "unperturbed" jet will be understood, for the purpose of this discussion, to describe the hypothetical reference condition in which the flow rate is sensibly constant over a circular section to some point downstream where a break occurs into an expanded fluctuating flow region. The most prominent characteristic of this fluctuating region in the reference state is the appearance of a succession of more or less well-formed discrete coaxial vortex rings. Each is formed at the tip of the cylindrical column by an outward roll-up of the cylindrical vortex sheet surrounding the jet until its strength is such as to detach itself and then permit another to form in the same way. After a steady state has been established, the time-average length of the jet remains constant and the sum of the strengths of the vortex circuits detached per unit of time is equal to the strength of a length of vortex sheet extruded from the nozzle in the same time. An unlimited expansion of the fluctuating region in real jets is prevented by the dissipative action of viscous forces and the idealized symmetrical structure never obtains because it is also unstable, though to a lesser degree than the jet, and is subject to everpresent perturbation tending toward random turbulence. This reference condition substantially describes the "sensitive" jet of the early literature.

If the reference jet is perturbed near its base by an impulse of some kind, the local disturbance so caused is carried downstream by the current of the jet.

A columnar flow of fluid is in unstable equilibrium. One of the manifestations of instability is the break of the unperturbed jet into a terminal field of discrete vortex circuits. A second consequence of instability is that the transported local disturbance caused by an impulse at the base, grows in magnitude as it travels. If the initiating impulse is small enough, its growth along the jet follows an exponential law and its influence on the terminal field is negligible. If it is large enough, the exponential increase may obtain for a certain distance along the jet after which a nonlinear disintegration process begins which may modify or dominate the terminal field, substituting in its place, a field which is more characteristic of the perturbation. If the perturbing force at the base is still larger, the whole jet column will show a pronounced structure and the terminal process of fragmentation will be radically changed.

The agency which produces the perturbation may be either external to the system, such as a vibration impressed on the nozzle, a sound wave, or air currents, or it may be a self-excited oscillation in the jet stream caused by a favorable nozzle contour or projections, and a proper flow rate. The presence in the stream of particles or other inhomogenesities also produces such effects.

The phenomena which occur within the range of amplitudes of waves on the jet resulting in exponential changes are the subject of this report. In this range the traveling disturbance may be treated by linear perturbation theory. The results of this theory may be used as a guide in qualitative reasoning concerning the phenomena observed at larger amplitudes. In particular, it is useful in speculation concerning the nature and granularity scale of the resulting fragmentation of the jet.

The procedure is to deduce and classify the orthogonal wave mode functions or mutually independent patterns of motion, each of which, when excited at any single frequency, is transmitted as a wave along the jet without change in form. Any mode, when excited by more than one frequency or when the flow is unsteady, gives rise to wave group phenomena, as the phase velocity varies with frequency. A central problem is to find the Fourier-Bessel expansion of an arbitrary initiating disturbance. This study is limited to the most prominent components of such expansion.

This report is principally aimed at providing a conceptual picture of a significant aspect of subsonic jet action, and to provide the basis for an experimental approach. While many diversions suggest themselves in this study, such material is reduced to a minimum consistent with its physical objective.

The divergence of experimentally observed phenomena from the corresponding idealized calculation is principally due to the omission of viscous forces in the theory. In a medium such as water or air, this is not so serious, for the lower modes of propagation, but that they may be readily recognized, and the theory usefully employed in their interpretation and study. The higher order fine-grained modes however may be largely obscured. To deal with the practical problem in this region a more difficult theoretical procedure is in prospect.

The early work on stability of jets is summarized in Rayleigh "Theory of Sound" Vol. II. A very interesting series of photographs and quantitative measurements on certain properties of jets related to the theory dealt with in this report, have been published by G. B. Brown.[1] As one illustration, which has been more recently worked out, of the effects of viscosity, reference is made to "Laminar Boundary Layer Oscillations, etc." by Schubauer and Skramstad.[2] While this work does not apply to the problem here considered, except indirectly as a possible perturbing mechanism within the nozzle, it gives a good illustration of what is probably in store for the extension of the theory to include viscous effects.

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[1] Proc.Phys.Soc. Lond. 47, (1935); 49 , (1937).

[2] Bur. Stand. report RP 1772, Feb. 1947.

2. Equations of Motion

In the idealized theory, a jet is supposed to consist of an irrotational flow in an incompressible fluid. In such a case any component of velocity and the pressure may be derived from a scalar potential, V, and the condition that the flow be incompressible is given by the Laplace equation:

$$\nabla^2 V = 0 \tag{1}$$

This will in general exclude the possibility of considering the action of viscous forces. If we find the function V (x,y,z,t) which satisfies this equation and the boundary conditions, the velocity is given by its negative gradient.

In the circular jet the symmetry is cylindrical so that the convenient form for (1) is:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2}$$

in which r,z,θ are the cylindrical coordinates. Being linear, a typical solution of this equation may be obtained as a product RHZT where each factor is a function of one variable, radius, angle, axial distance, and time, only. The particular form of solution of interest in this jet study is one representing waves and a typical term in such a solution is:

$$V = CR\,\exp(int - is\theta - imz) \tag{3}$$

in which case the differential coefficients with respect to t, θ , and z may be replaced by algebraic operators:

$$\frac{\partial}{\partial t} = in; \quad \frac{\partial}{\partial \theta} = -is; \quad \frac{\partial}{\partial z} = -im \tag{4}$$

Eq. (2) then reduces to a simpler form representing the typical case:

$$r^2 V'' + rV' - (s^2 - m^2 r^2)V = 0 \qquad (5)$$

where the primes represent differentiation with respect to r. In order to make V a single-valued function it will be seen from (3) that s must be an integer or zero.

This is Bessel's equation with a general solution:

$$V = AI_s(mr) + BK_s(mr) \qquad (6)$$

in which A and B are understood to contain the t, θ , z factors of (3) only.

The functions I_s and K_s are non-oscillating. I_s, for small values of the argument, is finite and increases exponentially for large values. K_s for small argument approaches infinite values and decreases exponentially for large argument.

Inside and outside the jet are two separate domains. Inside, the potential will be designated V_1-Wz, V_1 pertaining to any perturbation and -Wz the steady value of the unperturbed jet. W is the steady velocity in the direction z, of flow. Outside the jet the only potential is that of the perturbation. It will be designated V_2. To make the potential finite on the jet axis and at infinity we must have:

$$V_1 = AI_s(mr)\,; \qquad V_2 = BK_s(mr) \qquad (7)$$

The constants A and B are determined by conditions at the vortex sheet or bounding surface of the jet. These are that the normal component of velocity of fluid inside and outside are equal and that the pressure is continuous.

If the unperturbed radius of the jet is "a" and its small outward displacement at the point (r, θ, z) is q, when perturbed, then the instantaneous outward velocity of the sheet at this point is $\dot{q} = inq$. Let u, v, w represent the r, θ, and z components of fluid velocity of the perturbation which is superposed on W, the steady flow which is fixed in the z direction. The particle velocity components inside the jet are then u_1, v_1, $(w_1 + W)$ and outside, simply u_2, v_2, w_2. The small radial component of velocity of a particle just inside the jet is the first total time derivative of its instantaneous displacement q. In cylindrical coordinates this is given by the kinematic formula:

$$u_{1a} = \frac{dq}{dt} = \frac{\partial q}{\partial t} + u_1 \frac{\partial q}{\partial r} + \frac{v_1}{r}\frac{\partial q}{\partial \theta} + (w_1 + W)\frac{\partial q}{\partial z} \tag{8}$$

Since q is by definition, not a function of r, the second term disappears. If the displacement q were constant, the first term would disappear leaving the last two to represent the radial component of the resulting steady flow along the static displaced and oriented surface. The orientation is defined by its slopes $\frac{\partial q}{r\partial\theta}$ and $\frac{\partial q}{\partial z}$. Of these terms, all but the first and last containing the factor W are of second order and negligible.

$$u_{1a} = \frac{\partial q}{\partial t} + W\frac{\partial q}{\partial z} = i(n - mW)q \tag{9}$$

Similarly, the particle velocity just outside the vortex sheet is, to first order:

$$u_{2a} = \frac{\partial q}{\partial t} = inq \tag{10}$$

From (7) we obtain the particle velocities:

$$u_1 = -\frac{\partial V_1}{\partial r} = -mAI_s'(mr); \quad u_2 = -mBK_s'(mr) \tag{11}$$

where the primes refer to differentiation with respect to the whole argument, it being understood that m is a constant in this differentiation. When r = a these may be substituted in (9), (10).

$$-mAI_s'(ma) = i(n-mW)q \tag{12}$$

$$-mBK_s'(ma) = inq \tag{13}$$

These determine the ratio A/B.

$$\frac{AI_s'(ma)}{BK_s'(ma)} = \frac{n-mW}{n} \tag{14}$$

The general expression giving the hydrostatic pressure p, is

$$\begin{aligned} \frac{p}{\rho} &= \frac{\partial V}{\partial t} - \frac{1}{2}\left[u^2 + v^2 + (w+W)^2\right] \\ &= \frac{\partial V}{\partial t} - \frac{1}{2}\left[\left(\frac{\partial V}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial V}{\partial \theta}\right)^2 + \left(W - \frac{\partial V}{\partial z}\right)^2\right] \\ &= \frac{\partial V}{\partial t} + W\frac{\partial V}{\partial z} \quad \text{in the first order.} \end{aligned} \tag{15}$$

Outside the jet W = 0. We then have:

$$p_1 = i\rho(n-mW)V_1; \quad p_2 = in\rho V_2 \tag{16}$$

At the jet surface these must be equal:

$$\frac{AI_s(ma)}{BK_s(ma)} = \frac{n}{n-mW} \tag{17}$$

This, with (14) gives the secular equation relating the frequency n, the propagation constant or wave number m, and the order number s, which must be satisfied in order to satisfy the Laplace equation (2) and the wave form (3) simultaneously.

$$\frac{K_s I_s'}{K_s' I_s} = \frac{(n - mW)^2}{n^2} \tag{18}$$

For the purpose of study of this relation, it is convenient to define $x = ma$ and $g = na/W$. The first is then the wave number in units of the reciprocal readius of the jet. Physically, if m is real then $m = \frac{2\pi}{\lambda}$ and $x = \frac{2\pi a}{\lambda}$ or $m = 2\pi f/c$ and $x = 2\pi fa/c$ where c is the phase velocity. If on the other hand, n is real, it must be $2\pi f$ and $g = 2\pi fa/W$. Now $2\pi a/W$ is the interval of time taken by the jet to traverse the distance $2\pi a$ equal to its circumference. Thus g is the dimensionless frequency in units of $W/2\pi a$, a fixed characteristic of the jet.

$$\frac{K_s(x) I_s'(x)}{K_s'(x) I_s(x)} = \frac{(g - x)^2}{g^2} \tag{19}$$

In a study of the nature of the roots of this equation, it is convenient to modify its form. Two other useful forms may be obtained by use of the following recurrence formula:

$$x I_s' = s I_s + x I_{s+1} = -s I_s + x I_{s-1} \tag{20}$$

and

$$K_s' I_s = K_s I_s' - \frac{1}{x} \tag{21}$$

If we define B_s as follows:

$$B_s = K_s(sI_s + xI_{s+1}) = K_s(-sI_s + xI_{s-1}) \tag{22}$$

The secular equation may then also be written:

$$B_s(x) = \frac{(g-x)^2}{x(x-2g)} \tag{23}$$

Generally, either x or g may have an assigned complex value, after which (23) determines a complex value for the other. The simplest physical interpretation results when one or the other is chosen real.

3. Classification of Modes- Fluting Factor

The characteristic wave modes or propagation channels are identified by the order number "s". The surface of the jet is fluted, the pattern comprising s longitudinal ridges which may be parallel to the axis or spiral or a combination of both.

If an arbitrary periodic disturbance be impressed at some "drive" point on the jet, the result may be described as a complicated distribution of potential in a thin plane lamina passing through the jet perpendicular to its axis at that point. This potential distribution is then passed on to successive planes, mostly downstream, changing its distribution and amplitude as it travels. If this general disturbance be then broken down into single frequency components, each such component may be thought of as initiating a propagated wave in each of a series of independent channels or wave modes. Each such mode has its own characteristic potential distribution. The representative component elemental mode is denoted by the letter "s".

A specific solution for the s mode which is general enough to describe any steady state wave, consists of the sum of terms obtained from the typical wave form (3) by substitution in it of every combination of possible values of s, n, and m, as obtained from (23). Each such term also contains an arbitrary constant factor to be determined by further physical consideration.

Whatever the relative values of n and m might be (or x and g), the kinematic constant s is always a real integer or zero. If we define s as positive, the possible values are $0, \pm 1, \pm 2$ etc. The integer s is also the order number of the Bessel functions and since these are even functions of their orders there is no distinction between positive and negative orders. These circumstances permit of the isolation of s in an angular distribution factor.

If in (3) we write $R_s(mr)$ for R to represent either $I_s(mr)$ or $K_s(mr)$ depending on whether the potential inside or outside is intended, we may write a representative pair of terms of the complete steady state mode function as follows:

$$V = R_s(mr)(ae^{is\theta} + be^{-is\theta})\exp(int - imz) \qquad (24)$$

It will be noted that the Θ dependent factor of (24) is simply periodic with s periods per 2π angle. Redefine it as follows:

$$ae^{is\theta} + be^{-is\theta} = C H_s \qquad (25)$$

where H_s is a periodic function of unit amplitude to be defined explicitly and C, an external constant to specify magnitude. The most general case will be that in which a, b, C and H_s are complex. If we designate the left side of (25) by f, the square of its absolute value is:

$$f\bar{f} = a\bar{a} + b\bar{b} + a\bar{b}e^{2is\theta} + \bar{a}be^{-2is\theta} \qquad (26)$$

Let $a = r_1 (\exp(i\sigma_1)$ and $b = r_2(\exp(i\sigma_2)$ r_1, r_2, and σ_1, σ_2 being real positive numbers; (26) becomes

$$f\bar{f} = r_1^2 + r_2^2 + 2 r_1 r_2 \cos(\sigma_1 - \sigma_2 + 2s\theta) \tag{27}$$

When $\cos(\sigma_1 - \sigma_2 + 2s\theta) = 1$, corresponding to the angle $\theta_m = \frac{1}{2s}(\sigma_2 - \sigma_1)$

this expression takes the maximum value

$$f_m \bar{f}_m = (r_1 + r_2)^2 \tag{28}$$

If we now temporarily let C = 1 and normalize (28) to unit amplitude, r_1 and r_2 must satisfy the condition

$$r_1 + r_2 = 1 \tag{29}$$

This condition suggests the use of $r_1 = \cos^2 h$ and $r_2 = \sin^2 h$, where h is a single real assignable constant. These forms have the added advantage that as h is varied, each has a maximum value of unity and h may have any value whatever. Now reinstating C,

$$f = C\left(\cos^2 h e^{i(s\theta + \sigma_1)} + \sin^2 h e^{-i(s\theta - \sigma_2)}\right) \tag{30}$$

Let us examine the wave-functions corresponding to some specific value of h. If for example, h = 0 and $\sigma_1 = 0$, the coefficients of the exponentials in H_s are 1 and 0. If h = 0 and $\sigma_1 = \pi$, they are -1 and 0. If h = $\pi/4$ and $\sigma_1 = \sigma_2 = 0$, they are each 1/2 and the whole term is $\cos(s\theta)$.

And physical value may be described by a suitable choice of h, σ_1, σ_2.

It will be seen by examining (24) that two superposed spiral waves are represented and that σ_1 and σ_2 of the more explicit form (30) of the angular or fluting factor may serve to rotate these spirals about the jet axis by amounts which may be arbitrarily chosen to meet any requirements. Such latitude will in general be required in solving a specific quantitative problem; but for the physical picture, which is the object of this report, it is simplest to adopt an origin for the coordinates z and θ, for which these quantities are zero. The fluting factor then is:

$$H_{s0} = \cos^2 h e^{is\theta} + \sin^2 h e^{-is\theta} \qquad (31)$$

To recapitulate, the potential (24) written in terms of the fluting factor (30) or its specialized form (31) and the external constant C is now written:

$$V = C R_s(mr) H_s \exp(int - imz) \qquad (32)$$

It should be borne in mind that this is only a representative component part of a complete mode function, but that H_s is a factor common to all such components. The classification and description of the modes is based on the properties of H_s.

If a series of 2s apertures be disposed around the periphery of the nozzle and pulsations in alternately opposite phases injected into the jet stream through them, the s mode with $h = \pi/4$, principally will be excited at the pulsation frequency. With s apertures, all in the same phase, the s mode with $h = \pi/4$, and the $s = 0$ mode will be excited principally, both at the pulsation frequency. If with 2s apertures, alternate ones are actuated by a small steady air pressure, thus acting as "sources" while the others are subject to a small steady suction, thus acting as "sinks", and the whole aperture mount rotated at the rate, f per second, the s mode with $h = 0$ will be principally excited at the frequency sf. The same result may be obtained, in principle, by rotating a serrated mask with an aperture of the same average diameter as the jet, (provided conditions are not such as to produce "edge" tones).

The fluting factor H_s, is not as definite a function in the jet as in the analogous acoustic or electromagnetic systems. Physically this comes about because no real first order wave is transmitted tangentially on the jet, and only along it because a local disturbance is carried by the current. There is no elastic potential energy in the device, and while this phenomenon may be described in a wave language, it is not very closely analogous with waves in an elastic medium. Second order effects, such as those neglected at the boundaries (see Eqs. (8) and (15)) tend to introduce tangential waves.

The physical picture of the components of perturbation waves is clarified by examining their forms separately.

Constrictional waves, $s = 0$, $H_o = 1$. In this case all disturbances are axially symmetrical. This mode describes a wave of peristaltic or constrictional form traveling along the jet like beads on a string. It may be generated by a pressure fluctuation in the reservoir behind the nozzle or by a constrictional vibration of the periphery of the nozzle. The waves are easily made visible in a small jet of colored water discharging into clear water. To follow them by eye, the jet must be slow and the pulsations of a period such as to produce pulses spaced at a distance of the order of the jet diameter. The "beads" grow as they travel.

The growth in amplitude shows itself in an increasing pinching off of the successive "beads" which are one wave length apart. Each bead is surrounded by the vortex sheet of length λ, which, when separation has proceeded to a point, quickly rolls up into the core of a single self-propelled ring. The whole system has axial symmetry.

The secular equation applicable to this case is:

$$x I_1(x) K_0(x) = \frac{(x-g)^2}{x(x-2g)} \tag{33}$$

The equivalent of this for a two dimensional jet is given by Rayleigh (Theory of Sound, Vol. II) as:

$$\frac{\sinh x}{\cosh x} = \frac{(x-g)^2}{g^2} \tag{34}$$

<u>Transverse waves</u>, s = 1. For the standing wave form, h = $\pi/4$, in this case, $H_1 = \cos\theta$. This describes a transverse displacement of the jet section in the direction $\theta = 0$ without distortion of its circular form. The motion of the jet is then whiplike in the plane $\theta = 0, \pi$. Another similar independent

wave in the plane $\theta = \pm\frac{\pi}{2}$ may be produced and a combination in proper relative phase and amplitude gives a wave in the form of a spiral traveling downstream.

Waves of this kind are produced experimentally in one plane by a lateral vibration of the nozzle or by sound waves passing across the base of the jet. This is the wave generated acoustically in "sensitive" jets or flames which were popular in the latter part of the last century.

The end effects of this type of wave is clearly shown by photographs and quantitative measurements in two papers by G. B. Brown.[1] The transverse wave grows in amplitude as it travels down the jet until a point is reached where it is no longer able to make the sharp turns at the wave crests. Rupture there occurs and half wave sections of the jet are detached and thrown alternately to one side and then the other. Each section, as it leaves, may in the ideal, be regarded initially as a cylindrical column of length $\frac{\lambda}{2}$ surrounded by a vortex sheet. In accordance with the well-known properties of such a sheet, it quickly rolls outward and back on itself at the leading end and inward on the trailing end to form finally the single core of a self-propelled vortex ring. The succession of these rings and the angles of their trajectory are clearly brought out in Brown's photographs.

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[1] Loc. cit.

A secular equation for this mode is:

$$K_1(x)\left(I_1(x) + x\,I_2(x)\right) = \frac{(x-g)^2}{x(x-2g)} \tag{35}$$

The equivalent of this for the two dimensional jet (see Rayleigh) is:

$$\frac{\cosh x}{\sinh x} + \frac{(x-g)^2}{g^2} = 0 \tag{36}$$

<u>Second order mode</u>, $s = 2$. For the standing wave form, $h = \pi/4$, $H_2 = \cos(2\theta)$, A circular section of the jet is converted into an ellipse, which form is propagated downstream with its major and minor axes changing places each half wave length. This is the lowest order in which a distortion of the section takes place.

$$B_2(x) = \frac{(x-g)^2}{x(x-2g)} \tag{37}$$

Any order higher than $s = 1$ has no parallel in a two dimensional jet. Higher orders describe longitudinally fluted corrugations which, in the standing wave form, give $H_s = \cos(s\theta)$. The amplitude of this form along the jet is proportional to $\cos(s\theta)\cos(mz)$.

4. Corrugation and Propagation Modes

The secular equation (23) is to be regarded as one relation between n and m for a fixed s. The second relation required to determine both is arbitrary. Of the infinity of such arbitrary relations, two are most easily susceptible of physical interpretation. If, for the first of these, we define x as a real positive number we may easily solve for g in the quadratic (23). By reference to (3) it will be seen that in so doing, a sinusoidal "corrugation" of wave length $\lambda = \frac{2\pi}{m}$ extending along the full length of the jet is described and the calculation of g (or n) constitutes a determination of its change with time. In the second case we choose g to be a real positive number, i.e., we specify a real frequency $\frac{n}{2\pi}$ after which a calculation of x from (23), now to be regarded as transcendental, amounts to finding the "propagation constant" or "wave number," m, of a wave in the "steady state" (not in the hydrodynamical sense) as it travels along the jet. It is found that if one of these, g or x, is real the other must in general be complex. For describing "transient" behavior both are in general, complex.

The propagation modes (n real) are the ones which naturally occur in jets. They may easily be studied experimentally. A short discussion of corrugation modes, which cannot be studied experimentally, is included as it provides the stability criteria used by early writers. This also illustrates some more general properties of possible solutions of the jet equations and is suggestive of properties of transient behavior, the specific study of which is reserved for another time.

5. Corrugations

The solution of (23) for g gives:

$$g = x\left[(1-B_s) \pm i\sqrt{B_s(1-B_s)}\right] \tag{38}$$

By reference to a table of Bessel functions it will be seen that the quantity B_s for positive values of x ranges between the values 0 and 1/2, being usually 0.4 or 0.5 for all values of x except for B_0, which, as X = 0, becomes zero also. Both the real and imaginary parts of g tend usually to be somewhat less than x/2 but not equal to each other. Note in (38) that the real part of g has the same sign as x. We will not be concerned with possible negative values of x at this time.

Now by definition of g,

$$
\begin{aligned}
in = \frac{igW}{a} &= \frac{ixW}{a}\left[(1-B_s) \pm i\sqrt{B_s(1-B_s)}\right] \\
&= i\omega \pm \delta
\end{aligned}
\tag{39}
$$

From this, if one of these values be placed in (26) it appears that after being released, the corrugation moves downstream, i.e., in the direction z positive while its amplitude changes with time as it travels.

If we now reproduce (32) with the simplified notation (39)

$$
V = CR_s(mr)\,H_s \exp\left(i(\omega \pm i\delta)t - imz\right) \tag{40}
$$

A complete "steady state" corrugation potential mode function is therefore represented by two terms like this, each with an arbitrary constant and may be written:

$$
V = R_s(mr)\,H_s\left(C_1 e^{\delta t} + C_2 e^{-\delta t}\right)\exp\left(i\omega t - imz\right) \tag{41}
$$

This represents a wave of the fluting form H_s and wave length $\lambda = \frac{2\pi}{m}$ propagated downstream, i.e., in the direction of z positive, of frequency $f = w/2\pi$ and phase velocity $c = w/m$ with an amplitude changing with time in accordance with the parenthesis. There are therefore two components, one which increases without limit in amplitude with time and one which decreases. The values of these components depend on the initial conditions. At time $t = 0$,

$$V_0 = R_s(mr)\, H_s\, (C_1 + C_2)\, e^{-imz} \tag{42}$$

$$\dot{V}_0 = R_s(mr)\, \left((i\omega+\delta)\, C_1 + (i\omega-\delta)\, C_2\right) e^{-imz} \tag{43}$$

$$\frac{\dot{V}_0}{V_0} = \frac{(i\omega+\delta)\, C_1 + (i\omega-\delta)\, C_2}{C_1 + C_2} \tag{44}$$

We now describe a hypothetical experiment. Imagine an infinitely long jet encased in a rigid cylindrical sheath. The latter has corrugations which cause the jet to conform to the form factor H_s and $\exp(-imz)$. The potential inside the jet is given by (42) with $R_s = I_s$. Consequently the potential outside is $V_2 = 0$. As long as this sheath remained, the flow would be steady and $\dot{V} = 0$, that is, from (44):

$$C_2 = \frac{C_1(\delta + i\omega)}{\delta - i\omega} \tag{45}$$

If the sheath is suddenly released without at the same time imparting a velocity, the subsequent course of events is described by (41) with the value C_2 of (45) substituted. The initial amplitude of the two components will be equal but the decreasing component will be in phase advance over the other by the angle of $\tan^{-1}(\frac{2\omega}{\delta})$.

If we regard the condition which obtains after the transient effect $\exp(-\delta t)$ has subsided as descriptive of the natural free process, then the quantity δ is both an index of instability (for the two dimensional jet this is Rayleigh's index), from the term $\exp(\delta t)$ and a relaxation time - from the term $\exp(-\delta t)$. The relaxation time measures the time taken to build up the external field to the configuration characteristic of the freely growing wave which is independent of initial conditions.

If on the other hand we wish to initiate a free process at the start we must make $C_2 = 0$ by imparting an initial velocity at the time of removing the sheath such that from (44) $\dot{V}_0 = (i\omega + \delta)V_\infty$ that is, it must be in phase advance by $\tan^{-1}\frac{\omega}{\delta}$ over the displacement.

By the same reasoning it should, in principle be possible to provide an initial condition which would cause the corrugation to subside completely by imposing an initial velocity $\dot{V}_0 = -(\delta - i\omega)V_0$. This is of the same magnitude as that producing the free wave but in phase $\pi - 2\tan^{-1}(\frac{\omega}{\delta})$ relative to it. The impulse causing this effect would be such as to annul the internal field in a time measured by $\frac{1}{\delta}$.

It would thus appear that regardless of the relative values of C_1 and C_2 of (44), except for the critical case where C_1 is exactly zero, the increasing amplitude term must eventually predominate and the process end in disruption of the jet. If in an experiment such as that described, the imposition of an initial velocity were to be left to chance, there would be a vanishing probability of subsidence. The general process is then that after release, it hesitates for a time $1/\delta$, more or less, after which it proceeds on its unhampered career to destruction. This phenomenon has its counterpart in the more important progressive waves.

Fig. 1 gives corrugation phase velocities and stability or relaxation indices in convenient units as a function of the dimensionless wave number $x = ma = 2\pi a/\lambda$. They are described as follows:

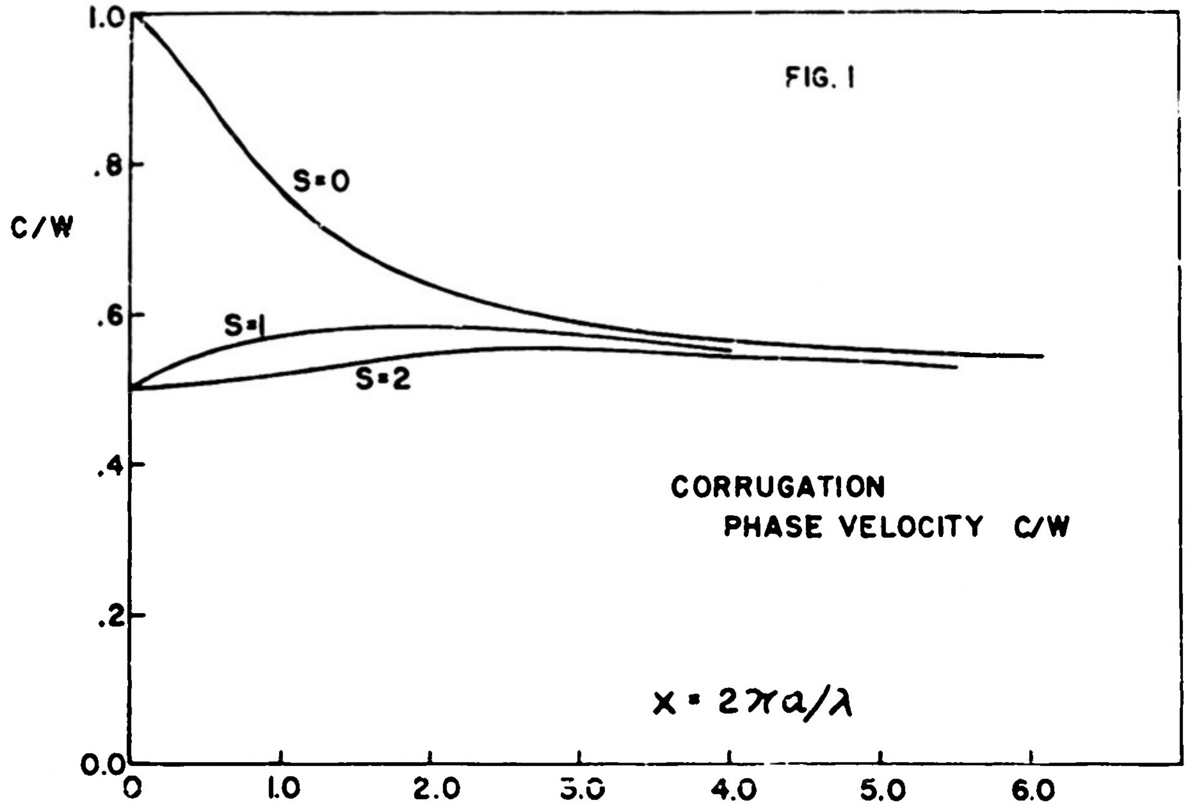
The "frequency", $\omega/2\pi$ associated with any one of these patterns, i.e., the frequency with which wave crests pass a given point is from (39)

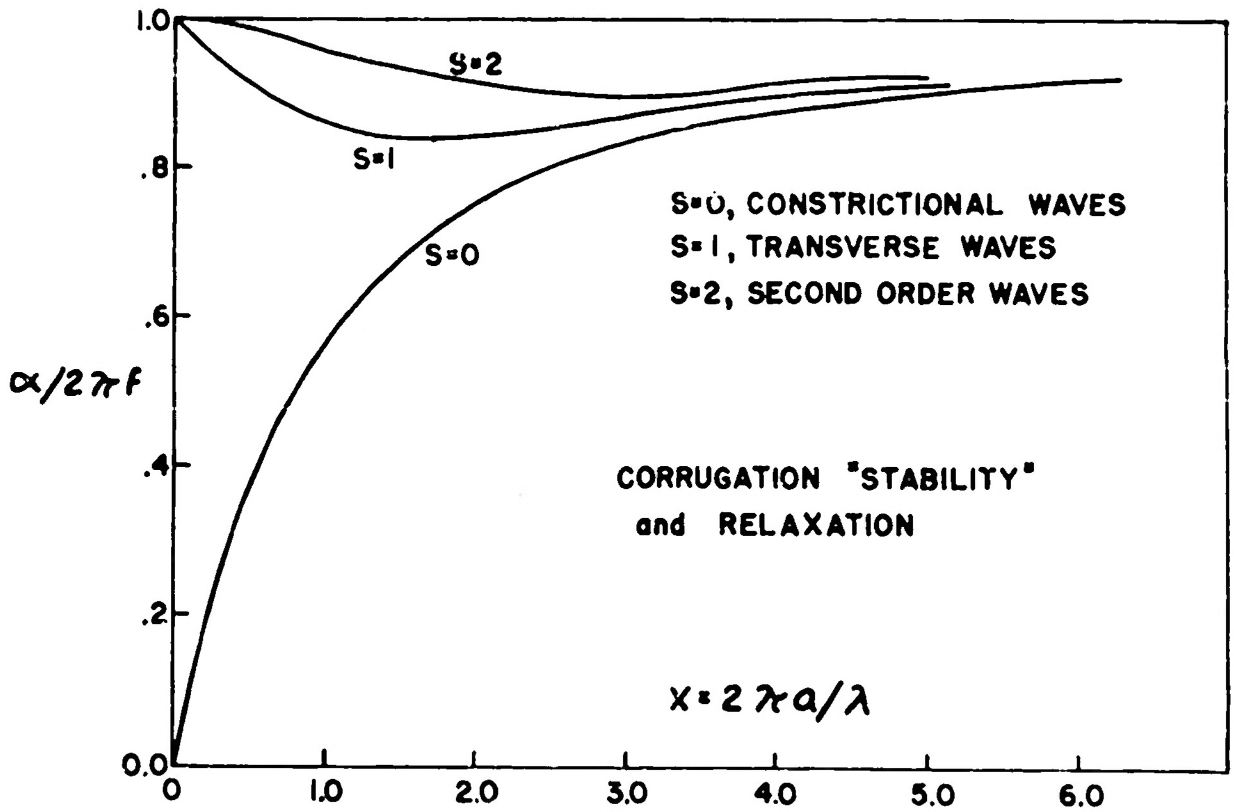
$$\frac{\omega}{2\pi} = \frac{W}{\lambda}(1 - B_s) \tag{46}$$

The corresponding phase velocity, $c = f\lambda$, is

$$c = W(1 - B_s) \tag{47}$$

This is plotted in the upper curves of Fig. 1 for the orders $s = 0$, 1, and 2 in units of W, the jet speed. With the exception of the long zero order waves, they all move with a speed a little greater than half the jet speed, i.e., a little faster than that of the vortex sheet. The higher the order


1.0
.8
.6
.4
.2
0.0
C/W
FIG. 1
S=0
S=1
S=2
CORRUGATION
PHASE VELOCITY C/W
X = 2πa/λ
0
1.0
2.0
3.0
4.0
5.0
6.0


1.0
.8
.6
.4
.2
0.0
α/2πf
S=2
S=1
S=0
S=0, CONSTRICTIONAL WAVES
S=1, TRANSVERSE WAVES
S=2, SECOND ORDER WAVES
CORRUGATION "STABILITY"
and RELAXATION
X = 2πa/λ
0
1.0
2.0
3.0
4.0
5.0
6.0

number, the more nearly they approach the speed of the vortex sheet. This hypothetical phenomenon then seems to be more characteristic of the vortex sheet than it is of the jet which seems not unreasonable if the jet is thought of in terms of a closely spaced succession of vortex rings.

A simple unit for plotting δ is ω .

$$\frac{\delta}{2\pi f} = \sqrt{\frac{B_s}{1-B_s}} \tag{48}$$

This quantity is plotted in the lower curves of Fig. 1 and it is seen to approach unity for all except the long wave or low frequency constrictional waves to which the jet is therefore relatively stable. When this quantity is unity the amplitude ratio of increase is e in one radian or exp(2π), about 1:500 per cycle. This amounts to an almost explosive increase in amplitude. It represents the limit which a real jet should approach as the viscosity approaches zero. Viscous forces have a strong stabilizing influence.

6. Wave Propagation Modes

Under this head we consider the description of waves propagated along the jet as a result of a perturbing force localized at some point or within a definite region on the jet. This is the type of wave always present to some small extent at least, in real jets. It is subject to direct experimental study.

The s mode will be considered separately to be driven at some point at the real frequency $f = n/2\pi$ which for the present will be assumed to be positive. It remains then to find the possible values of m from the secular equation (19) or (23).

It may be seen from the asymptotic expansions of K_s and I_s that when mod(x) is very large B_s approaches 1/2, for all orders. By using the series expansion it may be found that the same limit is approached when mod(x) = 0 for all orders except for s = 0, in which case it approaches the value zero. By placing these limiting values in (23) and solving for x it is found

$$x_\ell = g(1 \pm i) \qquad \text{and} \qquad x_{o\ell} = g$$

or

$$m_\ell = \frac{n}{W}(1 \pm i) \qquad \text{and} \qquad m_{o\ell} = \frac{n}{W} \tag{49}$$

To obtain values for intermediate frequencies requires a study of the general case (23). This has not been carried out. It will be noted that in the limiting cases one pair of complex conjugate roots is obtained with 45° arguments. The K and I therefore approach the Ker,kei and ber,bei functions. These, like the K and I are slowly varying, non-fluctuating and it seems quite likely that a proper study will show that the roots at intermediate frequencies will not be different from (49) by very large amounts. For the present we will assume conditionally that this is true. No quantitative use will be made of them. We then write in general

$$i m = i \beta \pm \alpha \tag{50}$$

where it is understood that both β and α are of the order n/W (except for $s = o$ when $n = 0$) but that they are probably not equal except at the limits. We may now write the complete propagation mode function which is analogous to (41) for corrugations.

$$V = \left[C_1 R_s((\beta + i\alpha) r) e^{\alpha z} + C_2 R_s((\beta - i\alpha) r) e^{-\alpha z} \right] H_s e^{int - i\beta z} \tag{51}$$

Notice that the same propagated fluting factor H_s that occurred in the corrugations, also occurs in this wave. The propagated wave is therefore the sum of two spiral waves channeled in opposite directions in the proportions determined by the nature of the generator at the drive point $z = 0$.

$$V_0 = \left[C_1 R_s((\beta + i\alpha) r) + C_2 R_s((\beta - i\alpha) r) \right] H_s e^{int} \tag{52}$$

The perturbed component of axial velocity $-\dfrac{\partial V}{\partial z}$ at this point is then:

$$w_0 = \left[-C_1 (\alpha - i\beta) R_s((\beta + i\alpha) r) + C_2 (\alpha + i\beta) R_s((\beta - i\alpha) r) \right] H_s e^{int} \tag{53}$$

where as before W_1 is obtained when R_s is replaced by I_s and W_2 by K_s. In this case it is to be noted that in order to preserve the cylindrical symmetry assumed so far, it is necessary that the jet issue from a flat baffle. This is essentially a semi-infinite system geometrically but (steady state) infinite in time in contrast with the corrugation in which the jet was required to be infinitely long but semi-infinite in time.

In this wave the phase velocity is nearly the same as W, the jet speed for all orders and is equal to it for both very low and high frequencies, i.e., for waves very long or very short compared with the jet diameter. This is in contrast with the corrugations which tend to move at about half this speed. The maximum departure from the limiting values of velocity occurs in the neighborhood of the region where the wave length is equal to the circumference of the jet, i.e., where g = 1 or mod(x) = 1.

Eq. (51) shows the wave to have two components in general, one which increases in amplitude as it moves along the jet (exp α z) and another which decreases at the same rate. Provided C_1 is not exactly zero, the growing component must ultimately predominate. The other component is then characteristic of the peculiarities of the wave generating device. The free wave which is characteristic of the jet alone is therefore:

$$V = C_1 R_s((\beta + i\alpha)r)e^{\alpha z} H_s e^{int - i\beta z} \qquad (54)$$

It will then be seen from (51), that there is a tendency for the amplitude of the disturbance to remain nearly constant for a distance measured in units of $1/\alpha$, after which the free rapid growth (54) obtains.

Since there are two arbitrary constants in (51) (in addition of course to the fluting constant "h" in H_g) it is necessary to specify the values of two quantities of the drive point to completely specify the wave. These may be any two convenient independent quantities such for example as the potential (52) and axial velocity (53) at that point. In this respect the jet is analogous to a thin flexible spring in which, it will be recalled, it is necessary to give the values of the force and couple, or of the displacement and its slope, at a point in order to completely describe the wave. In either case a sinusoidal wave obtains beyond a certain distance from the generator.

It appears from (52) and (53) that it is possible to adjust V_0 and w_o relatively in a way as to give the generator characteristics such that a free wave is radiated immediately from it ($C_2 = 0$) or such that nothing but a local disturbance ($C_1 = 0$) is created. Likewise it is possible, in principle, to adjust the generator ($C_2 >> C_1$) in such a way as to produce a large local disturbance near the generator, a very low amplitude wave following it and then a rapid growth at a distance.

The quantity α is also in the nature of a stability index to be compared with δ of (39). It is also in the nature of a relaxation index since it is a measure of the distance the wave must travel in order for the field configuration to adjust itself to that of the free wave which is characteristic of the jet alone. The time taken for the wave to travel a distance $1/\alpha$ is $1/c\alpha \simeq \tau$ and since c is of the order [illegible] and

ω of the order n/W, τ is of the order $1/n$, or $1/\tau$ is of the order n while δ is of the order [illegible]. The stability of the progressive wave is about half that of the corrugation.

7. <u>Vorticity and Fragmentation</u>

The surface of the unperturbed ideal jet is a cylindrical vortex sheet of uniform strength or "circulation" W per unit length. Circulation is measured by the line integral of velocity around a closed path of unit length, lying along the direction of flow and including in its area a unit length of the vortex sheet. This circulation is the same as that due to an average vorticity, or curl of the vector velocity, distributed over the area of the path multiplied by its area. It is sometimes convenient to think of this shear layer, which in idealized theory has no thickness, as a layer of infinitesimal thickness τ with vorticity (density) ω so that the total vortex strength per unit length is $\omega\tau = W$. With this conception the layer contains fluid in actual rotation on which that on both sides rolls. In this case the jet is enclosed by actual vortex filaments which have all the usual properties of such filaments in the non-viscous thermodynamically-inert medium which we have assumed. Since there are no external rotational forces acting on the medium, the substance of the vortex layer always contains the same particles and none can be added. It may be distorted, stretched, attenuated or thickened like a membrane, but unlike a membrane it can be broken only along the line of a vortex filament. No filament may be broken. Thus the only way it is possible for the ideal jet to break is transversely.

We may presume without a desirable proof that the same properties hold in the limit of the vortex sheet. Thus as long as we disregard viscosity, the jet may break only into cylindrical sections which, under the influence of perturbation which causes such fragmentation, may be individually distorted in any way as long as filaments are not broken. Experiments have been described above, which show this fragmentation in the case of orders s = o and 1.

Experiments are always conducted with a viscous medium. Although, in such systems, there always remains a tendency toward a behavior as above described, there are some qualitative and quantitative new properties introduced by viscous forces which it is desirable to bear in mind. These forces provide a means for vorticity to diffuse like heat from a filament, thus losing rotation and imparting it to neighboring fluid. Neighboring circuits may, under favorable circumstances, fuse to form larger circuits. Again, under proper circumstances, a single circuit may be broken into two or more circuits.

The equilibrium configuration of an isolated circuit is a circular ring. Under the influence of neighboring circuits, and particularly when fusion or fragmentation occurs, each circuit exists in what may be described as a state of "strain" and "excited" vibration. Many of these properties may be easily demonstrated with apparatus for making "smoke" rings and have been treated theoretically by Kelvin and others in the 1870's and 80's. A field of such circuits constitutes a "coupled" system, having modes of vibration characteristic of the system. The situation is in some degree logous with an elastically coupled system. Such a field is in general not an equilibrium configuration however and the interaction of its "vibrations" is pre-eminently nonlinear. The analytical treatment of such fields, except statistically, is impracticable except possibly in a few academic cases. It is a very long step between the perturbation and its resulting field. Any speculation regarding the structure of the fluctuating field so determined from a simple perturbation mode, should therefore be considered at most as a contribution toward clarifying certain pertinent questions and not generally as a prediction of what takes place. Any relation between perturbation and terminal field must be determined by suitably planned future experiment.

By analogy with the vibrations of an isolated ring, the zero and first order modes of the jet may be thought of as "unexcited" modes since, in the ideal, the section remains circular. Similarly the second and higher order mod may be thought of as "excited," since, as is the case of the vibrating ring, the periphery is distorted. The vorticity is then concentrated locally in certain regions of the sheet. This is a favorable condition, under the action of viscous forces, for finer grained fragmentation, quicker decay of the pattern, and for fusion or fragmentation of individual circuits.

When the jet is perturbed, the fluid in and around it acquires the additional small velocity components u, v, w. There is then superposed on the steady vortex strength W of the sheet, two components.

$$\omega_\theta = w_{1a} - w_{2a} = \frac{\partial}{\partial z} \cdot (V_{2a} - V_{1a}) \tag{55}$$

$$\omega_z = v_{1a} - v_{2a} = \frac{\partial}{a\,\partial\theta} (V_{2a} - V_{1a}) \tag{56}$$

The vortex strength is a vector, mostly directed to the θ direction, having the components $W + \omega_\theta$ and ω_z. Since V contains the factor H which consists of a whole number, s, of periods around the jet, the average values around the periphery of these components are W and 0. The vortex lines are distorted circles lying in the cylindrical surface (to first order).

The differential equation of the vortex lines in the sheet are then:

$$\frac{dz}{\omega_z} = \frac{a\,d\theta}{\omega_\theta} \tag{57}$$

and their equation $z = f(\theta)$, is found by integration. To find these lines, the real or "in phase" components of ω_z and ω_θ may be used.

Ordinarily these curves are not of very simple form. For example, for the s corrugation wave (see Eq. (41)) of the form $H_s = \cos(s\theta)$ at time $t = 0$ the equation is found to be:

$$mz \tan(mz) + \log|\cos(s\theta)| = 0 \tag{58}$$

The motion of the vortex sheet may be obtained from (9), (10), (11).

$$\dot{q} = -\left(\frac{\partial V_2}{\partial r}\right)_{r=a} = -\frac{\partial V_{2a}}{\partial a} \tag{59}$$

$$W\frac{\partial q}{\partial z} = \frac{\partial}{\partial a}\left(V_{2a} - V_{1a}\right) \tag{60}$$

From these and (51), (52) it appears that there is no simple correspondence between vortex strength and motion of the vortex sheet unless the specific forms of the potentials are chosen. The relative values of the three velocity components, u, v, w, are not the same in corrugation and propagation waves. For comparison the corrugation and propagation potentials for free waves obtained from (41) and (51) are respectively:

$$V = C R_s(mr) H_s e^{\delta t} \exp(i\omega t - imz) \tag{61}$$

$$V = C R_s((i\beta + i\alpha)r) H_s e^{\alpha z} \exp(int - i\beta z) \tag{62}$$

in which it will be recalled, $\lambda = \frac{2\pi}{m}$ is the corrugation wave length and $\omega/2\pi$ the resulting frequency while $n/2\pi$ is the impressed drive frequency with $2\pi/\beta$ the resulting wave length for the propagated wave. δ is of about the same magnitude as ω and α is about the same as β except, in both cases, for s = o in the low frequency range.